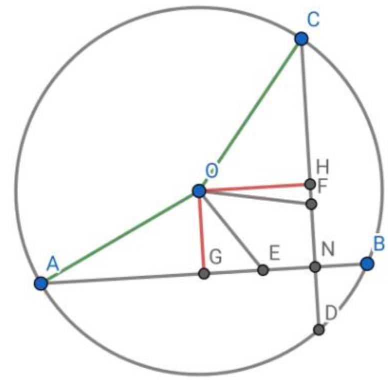
01.01.2024 Cash Award Math Rider : I Prize Winner Mrs. Madhumitha's Solution



Given :

1. O is the centre of circle.
2. AB & CD are two chords perpendicular to each other.
3. E & F are points on AB & CD respectively such that NB=NE & ND =NF

⟹ that E is the orthocentre of ΔADC. & F is the orthocentre of ΔABC

[From Novelty 3 of orthocentre in ΔABC inscribed in a circle. AD, the altitude to BC is produced to meet the circle at P. BE is altitude to CA & C is orthoccentre then OD= DP]

Construction:

Drop a perpendicular from O to AB & another perpendicular from O to DC which meets AB at G and DC at H.

Join OA & OC

OA = OC = r [radius of circle]

To Prove that $OE^2 - OF^2 = 2\,(NE^2 - NF^2)$

In ΔOGE, $OE^2 = OG^2 + GE^2$ ---------------------- (1)

In ΔOHF, $OF^2 = OH^2 + HF^2$ --------------------------(2)

$OE^2 - OF^2 = (OG^2 + GE^2) - (OH^2 + HF^2)$ ---------------- (A)

In ΔOAG, $OG^2 = OA^2 - AG^2 = r^2 - AG^2$

In ΔOHC, $OH^2 = OC^2 - CH^2 = r^2 - CH^2$ ⟶ (3)

Sub (3) in (A) becomes

$OE^2 - OF^2 = r^2 - AG^2 + GE^2 - r^2 + CH^2 - HF^2$ -----------(B)

$= GE^2 - AG^2 + CH^2 - HF^2$ ----------------------- (B)

GE = GB - EB , HF = HD - FD.

[Here, G is midpoint of AB & H is midpoint of CD.

$\because OG \perp AB, OG \perp CD$.

Perpendicular drawn from centre to chord bisect the chord]

Now (B) becomes

$OE^2 - OF^2 = (GB - EB)^2 - AG^2 + CH^2 - (HD - FD)^2$

[∵ GB = AG & CH =HD]

= $(GB - EB)^2$-GB^2 + HD^2 − $(HD - FD)^2$

= $(GB - EB - GB)$ $(GB - EB + GB)$+(HD-HD+FD) (HD+HD-FD)

=(2GB-EB) (-EB) +(FD) (2HD-FD)

=(2GB-2EN) (-2EN) +(2FN) (2HD-2FN)

=(AB-2EN) (-2EN) +2FN (CD-2FN)

=-2 AB x FN + $4EN^2$+2FN x CD - $4FN^2$

=4$[EN^2 - FN^2]$+ 2FN x CD - 2AB x EN

=2$[EN^2 - FN^2]$ + $2EN^2$-$2FN^2$ − $2FN$ x CD - 2 AB x EN -------- (C)

T.P.T

$OE^2 - OF^2$ = 2$[EN^2 - FN^2]$

It is enough to prove that

$2EN^2 - 2FN^2 +$ FN x CD - 2 AB x EN = 0

By the Novelty 8 of orthocentre (Page 24)

EN = $\frac{DN \times NC}{AN}$ & $FN = \frac{AN \times NB}{CN}$ ---------------------------- (4)

Also when two chords AB & CD intersect each other at N.

AN x NB = CN x ND -------------------- (5)

Consider :

$2EN^2 - 2FN^2 +$ $2FN$ x CD - 2 AB x EN

= 2 EN [EN - AB] + 2FN[CD-FN]

by (4)

= 2 $\frac{DN \times NC}{AN}$ [EN-AN-NB] + 2$\frac{AN \times NB}{CN}$ [CN +ND - FN]

(∵ $AB = AN + NB$, $CD = CN = ND$)

Also EN = NB & FN = ND (given)

= 2 $\frac{DN \times NC}{AN}$ x -AN +2$\frac{AN \times NB}{CN}$ x CN

= -2 DN x NC + 2 AN XNB by (5)

= -2 AN x NB + 2AN x NB

= 0

Hence $OE^2 - OF^2 = 2\ (EN^2 - FN^2)$

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